View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

ACADEMIC PRESS
Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®
J. Differential Equations 194 (2003) 287–299
Journal of Differential Equations
http://www.elsevier.com/locate/jde

# On a differential equation characterizing Euclidean spheres

Eduardo García-Río,[a,*,1] Demir N. Kupeli,[b] and Bülent Ünal[c]

[a] Department of Geometry and Topology, Faculty of Mathematics, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain
[b] Bosna-Hersek Caddesi, No:35/B-10, Emek Mahallesi, 06510 Ankara, Turkey
[c] Department of Mathematics, Atilim University, Incek, 06836 Ankara, Turkey

Received July 4, 2002; revised May 29, 2003

## Abstract

A characterization of Euclidean spheres out of complete Riemannian manifolds is made by certain vector fields on complete Riemannian manifolds satisfying a partial differential equation on vector fields.
© 2003 Elsevier Inc. All rights reserved.

*MSC:* 53C99; 58J99

*Keywords:* Second covariant differential; Laplacian; Conformal vector field; Affine conformal vector field; $k$-Nullity vector field; Projective vector field; Euclidean sphere

## 1. Introduction

In analysis, one often looks for existence (and possibly uniqueness) of a solution of a differential equation on a specified domain. Conversely, in geometry, one might consider the question of the existence of a domain (or manifold) that supports a solution of a specified differential equation. For example, given a domain in the plane, one seeks a solution to the Dirichlet problem. Conversely, assuming the existence of a solution of a second-order partial differential equation with

*Corresponding author.
*E-mail addresses:* xtedugr@usc.es (E. García-Río), ndkupeli@superonline.com (D.N. Kupeli), bulentunal@mail.com (B. Ünal).
[1] Partially supported by a project BFM2003-02949 (Spain).

0022-0396/$ - see front matter © 2003 Elsevier Inc. All rights reserved.
doi:10.1016/S0022-0396(03)00173-6

overdetermined boundary conditions, one might ask what this implies about the geometry of the domain. This may be considered as an analytic characterization (or representation) of a manifold by a differential equation if this manifold serves as a unique domain for this differential equation to possess a nontrivial solution in a certain class of manifolds. In the literature, some characterizations of rank-one symmetric Riemannian manifolds by differential equations can be found. For example, some known characterizations of Euclidean spheres, complex projective spaces and quaternionic projective spaces by differential equations can be found in [1,2,6,10–12,14,15], and also a survey of these papers can be found in [5].

It seems that one of the most significant examples of such a characterization of Euclidean spheres is a result of Obata [11], that is, a necessary and sufficient condition for a connected, complete Riemannian manifold $(M, g)$ to be isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$, $\lambda > 0$, is the existence of a nonconstant function $f$ on $M$ satisfying the (tensorial) differential equation $H_f + \lambda f g = 0$, where $H_f$ is the Hessian form of $f$ on $(M, g)$. In other words, the differential equation $H_f + \lambda f g = 0$, $\lambda > 0$, on a connected, complete, Riemannian manifold $(M, g)$ has a nontrivial solution if and only if its domain $(M, g)$ is the Euclidean sphere of radius $1/\sqrt{\lambda}$. Also, in this particular example, on the domain-connected, complete Riemannian manifolds $(M, g)$, the differential equation $H_f + \lambda f g = 0$, $\lambda > 0$, can be considered as an analytic characterization (or representative) of Euclidean spheres. In this paper, we state another differential equation (which is "equivalent" to the above Obata's equation) on connected, complete Riemannian manifolds $(M, g)$ characterizing Euclidean spheres by the existence of a nontrivial solution to this equation (see Theorem 3.5). More precisely, we show that, a necessary and sufficient condition for a connected, complete $n(\geqslant 2)$-dimensional Riemannian manifold to be isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$, $\lambda > 0$, is the existence of a nonzero vector field $Z$ on $(M, g)$ satisfying the differential equation $(\nabla\nabla Z)(\cdot\,,\cdot) + \lambda g(Z, \cdot)(\cdot) = 0$ on $(M, g)$, where $\nabla\nabla Z$ is the second covariant differential of $Z$. Hence, in the class of domain-connected, complete Riemannian manifolds $(M, g)$, the differential equation $(\nabla\nabla Z)(\cdot\,,\cdot) + \lambda g(Z, \cdot)(\cdot) = 0$, $\lambda > 0$, also serves as an analytic characterization (or representative) of Euclidean spheres. (See Remarks 3.7 and 3.8 for the stand of this differential equation among the other known differential equations characterizing Euclidean spheres). We also analyze the case $\lambda \leqslant 0$ for the differential equation $(\nabla\nabla Z)(\cdot\,,\cdot) + \lambda g(Z, \cdot)(\cdot) = 0$ on a connected, complete Riemannian manifold $(M, g)$. Yet we see that, in the case of $\lambda \leqslant 0$, this differential equation is not that deterministic about its domain connected, complete Riemannian manifold as in the case of $\lambda > 0$.

## 2. Preliminaries

Here, we briefly state the main concepts and definitions used throughout this paper.

Let $Z$ be a vector field on an $n$-dimensional Riemannian manifold $(M,g)$ with Levi–Civita connection $\nabla$. The second covariant differential $\nabla\nabla Z$ of $Z$ is defined by

$$(\nabla\nabla Z)(X,Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z,$$

where $X, Y \in \Gamma TM$. We define the Laplacian $\Delta Z$ of $Z$ on $(M,g)$ to be the trace of $\nabla\nabla Z$ with respect to $g$, that is,

$$\Delta Z = trace\, \nabla\nabla Z = \sum_{i=1}^{n} (\nabla\nabla Z)(X_i, X_i),$$

where $\{X_1, \dots, X_n\}$ is a local orthonormal frame for $TM$.

Also, if $Z$ is a vector field on a Riemannian manifold $(M,g)$ then the affinity tensor $L_Z\nabla$ of $Z$ is defined by

$$(L_Z\nabla)(X,Y) = L_Z \nabla_X Y - \nabla_{L_Z X} Y - \nabla_X L_Z Y,$$

where $L_Z$ is the Lie derivative with respect to $Z$ and $X, Y \in \Gamma TM$ (see, for example [13, p. 109]). We define the tension field $\square Z$ of $Z$ on $(M,g)$ to be the trace of $L_Z\nabla$ with respect to $g$, that is,

$$\square Z = trace\, L_Z\nabla = \sum_{i=1}^{n} (L_Z\nabla)(X_i, X_i),$$

where $\{X_1, \dots, X_n\}$ is a local orthonormal frame for $TM$.

By a straightforward computation, it can be shown by using the torsion-free property of $\nabla$ that

$$(L_Z\nabla)(X,Y) = (\nabla\nabla Z)(X,Y) + R(Z,X)Y$$

(see [13, p. 110]) and hence,

$$\square Z = \Delta Z + \widehat{Ric}(Z),$$

where $R$ is the curvature tensor of $(M,g)$, $\widehat{Ric}$ is the Ricci operator of $(M,g)$ and $X, Y \in \Gamma TM$. (also see [16, p. 40]).

## 3. A characterization of Euclidean spheres

First we state an elementary lemma to be used in the proof of the main result of this paper.

**Lemma 3.1.** *Let $(M,g)$ be an $n$-dimensional Riemannian manifold and $\lambda \in \mathbb{R}$. If $Z$ is a vector field on $(M,g)$ satisfying the equation $(\nabla\nabla Z)(X,Y) + \lambda g(Z,X)Y = 0$ for all $X, Y \in \Gamma TM$, then $\nabla div\, Z = -n\lambda Z$, and hence, $\nabla\nabla div\, Z = -n\lambda \nabla Z$, where $\nabla\nabla div\, Z$ is the Hessian tensor of $div\, Z$.*

**Proof.** Let $\{X_1, \ldots, X_n\}$ be an adapted orthonormal frame near $p \in M$, that is, $\{X_1, \ldots, X_n\}$ is an orthonormal frame in $TM$ with $(\nabla X_i)_p = 0$ for $i = 1, \ldots, n$, and let $X \in \Gamma TM$. Then at $p \in M$,

$$\begin{aligned} g(\nabla div\, Z, X) &= X\, div\, Z = \sum_{i=1}^{n} g(\nabla_X \nabla_{X_i} Z, X_i) \\ &= \sum_{i=1}^{n} g((\nabla\nabla Z)(X, X_i), X_i) = -\lambda \sum_{i=1}^{n} g(g(Z, X)X_i, X_i) \\ &= -n\lambda g(Z, X). \end{aligned}$$

Hence, it follows that $\nabla div\, Z = -n\lambda Z$. $\square$

**Remark 3.2.** Let $(M, g)$ be a Riemannian manifold and $\lambda \in \mathbb{R}$. A vector field $Z$ on $(M, g)$ satisfying $R(X, Y)Z = \lambda[g(Z, Y)X - g(X, Z)Y]$ for all $X, Y \in \Gamma TM$ is called a $\lambda$-nullity vector field on $(M, g)$ (see Proposition 3.3). That is, $Z$ is a nullity vector field with respect to the curvature-like tensor field $F(X, Y)W = R(X, Y)W - \lambda[g(W, Y)X - g(X, W)Y]$ on $(M, g)$ (see [14, Sections 2 and 4]). In particular, if there exists a nonzero $\lambda(\neq 0)$-nullity vector field $Z$ on a Riemannian manifold $(M, g)$ then $(M, g)$ is irreducible (see [3,7,14] and the references therein for details). Also note that, a vector field $Z$ on a Riemannian manifold $(M, g)$ is called affine conformal if $L_Z g = 2\phi g + \Omega$, where $\phi$ is a function on $M$ and $\Omega$ is a parallel, symmetric $(0, 2)$-tensor field on $(M, g)$. It is known that, if a Riemannian manifold $(M, g)$ admits such an $\Omega \neq cg$, where $c \in \mathbb{R}$, then $(M, g)$ is locally reducible (see [4, p. 142]).

**Proposition 3.3.** *Let $(M, g)$ be a connected, $n(\geqslant 2)$-dimensional Riemannian manifold and $Z$ be a nonzero vector field on $(M, g)$. If $Z$ satisfies the equation $(\nabla\nabla Z)(X, Y) + \lambda g(Z, X)Y = 0$, $\lambda \neq 0$, for all $X, Y \in \Gamma TM$, then*

(a) $\Delta Z = -\lambda Z$ *on* $(M, g)$,
(b) $R(X, Y)Z = \lambda[g(Z, Y)X - g(X, Z)Y]$ *for all* $X, Y \in \Gamma TM$ (*and hence*, $\widehat{Ric}(Z) = \lambda(n-1)Z$),
(c) $\nabla Z = \frac{div\, Z}{n} id$, (*and hence*, $Z$ *is a conformal vector field on* $(M, g)$),

**Proof.** (a) Obvious.

(b) Let $X, Y \in \Gamma TM$. Then,

$$\begin{aligned} R(X, Y)Z &= (\nabla\nabla Z)(X, Y) - (\nabla\nabla Z)(Y, X) \\ &= \lambda[g(Z, Y)X - g(X, Z)Y]. \end{aligned}$$

(c) First note that, since $Z$ is nonzero, it follows from Lemma 3.1 that $div\, Z$ is nonconstant and $\nabla\nabla div\, Z = -n\lambda \nabla Z$. Hence, $\nabla Z$ is self-adjoint and can be written as $\nabla Z = \frac{div\, Z}{n} id + \sigma$, where $\sigma$ is the traceless self-adjoint part of $\nabla Z$. Now we show that $\sigma = 0$ by using Lemma 3.1 and Remark 3.2. For this, first we show that $\nabla \sigma = 0$, that is $\sigma$ is parallel on $(M, g)$. Let $X, Y \in \Gamma TM$. Then by Lemma 3.1,

$$\begin{aligned}(\nabla\sigma)(X, Y) &= (\nabla\nabla Z)(X, Y) - \frac{1}{n} g(\nabla div\, Z, X)Y \\ &= (\nabla\nabla Z)(X, Y) + \lambda g(Z, X)Y \\ &= 0.\end{aligned}$$

Thus, $\sigma$ is parallel on $(M, g)$ and hence, $Z$ is an affine conformal vector field on $(M, g)$, that is,

$$(L_Z g)(X, Y) = 2\frac{div\, Z}{n} g(X, Y) + 2g(\sigma(X), Y)\ (= 2g(\nabla_X Z, Y))$$

for all $X, Y \in \Gamma TM$. Now, if $\sigma \neq 0$ then since $\sigma \neq c\, id$, where $c \in \mathbb{R}$, it follows from Remark 3.2 that $(M, g)$ is locally reducible. But this conflicts with the fact that $Z$ is a nonzero $\lambda(\neq 0)$-nullity vector field on $(M, g)$, that is, there is an open submanifold $(U, g)$ of $(M, g)$ which is irreducible by Remark 3.2. Thus, $\sigma = 0$ and it follows that $\nabla Z = \frac{div\, Z}{n} id$ (and hence $Z$ is a conformal vector field (see [13, p. 173])). $\square$

We also have the following two converses of the above proposition.

**Proposition 3.4.** *Let $(M, g)$ be a connected, $n(\geqslant 2)$-dimensional Riemannian manifold and $Z$ be a nonzero vector field on $(M, g)$. Then, $Z$ satisfies the equation $(\nabla\nabla Z)(X, Y) + \lambda g(Z, X)Y = 0$, $\lambda \in \mathbb{R}$, for all $X, Y \in \Gamma TM$, provided that either,*

(a) *$R(X, Y)Z = \lambda[g(Z, Y)X - g(X, Z)Y]$ for all $X, Y \in \Gamma TM$,*
(b) *$\nabla Z = \frac{div\, Z}{n} id$ on $(M, g)$*

*or*

(c) *$\Delta Z = -\lambda Z$ on $(M, g)$,*
(d) *$R(X, Y)Z = \lambda[g(Z, Y)X - g(X, Z)Y]$ for all $X, Y \in \Gamma TM$,*
(e) *$Z$ is a conformal vector field on $(M, g)$ (that is, $L_Z g = 2\frac{div\, Z}{n} g$).*

**Proof.** Suppose $Z$ satisfies (a) and (b). Then it follows from $\nabla Z = \frac{div\, Z}{n} id$ that $(\nabla\nabla Z)(X, Y) = \frac{1}{n} g(\nabla div\, Z, X)Y$ for all $X, Y \in \Gamma TM$, and hence $\Delta Z = \frac{1}{n}\nabla div\, Z$. Also, since $Z$ is a conformal vector field,

$$(L_Z\nabla)(X, Y) = \frac{1}{n}[g(\nabla div\, Z, Y)X + g(X, \nabla div\, Z)Y - g(X, Y)\nabla div\, Z]$$

for all $X, Y \in \Gamma TM$ (see [16, p. 46]) and since

$$(L_Z\nabla)(X, Y) = (\nabla\nabla Z)(X, Y) + R(Z, X)Y$$

for all $X, Y \in \Gamma TM$ (see Preliminaries), we obtain

$$(\nabla\nabla Z)(X, Y) = -\lambda[g(X, Y)Z - g(Y, Z)X] + \frac{1}{n}[g(\nabla div\, Z, Y)X + g(X, \nabla div\, Z)Y - g(X, Y)\nabla div\, Z]$$

by using the assumption that $Z$ is a $\lambda$-nullity vector field. Also, by taking the trace of this equation with respect to $g$ and then using $\Delta Z = \frac{1}{n}\nabla div\, Z$, we obtain that, $\nabla div\, Z = -n\lambda Z$. Hence, $(\nabla\nabla Z)(X, Y) = -\lambda g(Z, X)Y$ for all $X, Y \in \Gamma TM$.

Now suppose $Z$ satisfies (c)–(e). Then, as in the above case, we obtain

$$(\nabla\nabla Z)(X, Y) = -\lambda[g(X, Y)Z - g(Y, Z)X] + \frac{1}{n}[g(\nabla div\, Z, Y)X + g(X, \nabla div\, Z)Y - g(X, Y)\nabla div\, Z].$$

Also, by taking the trace of this equation with respect to $g$ and then using $\Delta Z = -\lambda Z$, we obtain that, $\nabla div\, Z = -n\lambda Z$. Hence, $(\nabla\nabla Z)(X, Y) = -\lambda g(Z, X)Y$ for all $X, Y \in \Gamma TM$. $\square$

Now we will prove the main result of this paper in the following theorem.

**Theorem 3.5.** *Let $(M, g)$ be a connected, complete $n(\geqslant 2)$-dimensional Riemannian manifold. Then, a necessary and sufficient condition for $(M, g)$ to be isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$, $\lambda > 0$, is the existence of a nonzero vector field $Z$ on $(M, g)$ satisfying the equation*

$$(\nabla\nabla Z)(X, Y) + \lambda g(Z, X)Y = 0$$

*for all $X, Y \in \Gamma TM$.*

**Proof.** Here we prove only the sufficient condition of this theorem. The example of the necessary condition is given in the remark below (see Remark 3.6). To prove the sufficient condition, we use a result of Obata ([11, Theorem A]). That is, we show that there exists a nonconstant function $f$ on $M$ satisfying the equation $H_f + \lambda f g = 0$, $\lambda > 0$, where $H_f$ is the Hessian form of $f$ on $(M, g)$. Clearly, by Lemma 3.1 and Proposition 3.3, we can choose this function $f$ as $div\, Z$ and the claim follows. $\square$

**Remark 3.6.** Now, we give an example of a vector field $Z$ on an $n(\geqslant 2)$-dimensional Euclidean sphere $(\mathbb{S}^n(r), \tilde{g})$ of radius $r = 1/\sqrt{\lambda}$, $\lambda > 0$, which satisfies the equation $(\tilde{\nabla}\tilde{\nabla}Z)(X, Y) + \lambda\tilde{g}(Z, X)Y = 0$ for all $X, Y \in \Gamma T\mathbb{S}^n(r)$, where $\tilde{\nabla}$ is the Levi–Civita connection of $(\mathbb{S}^n(r), \tilde{g})$ (see [13, p. 117] for details). Let $\chi : \mathbb{S}^n(r) - \{south\, pole\} \to \mathbb{R}^n$

be the stereographic projection and $Z$ be a vector field on $\mathbb{R}^n$ defined by $Z_p = (p, p)$. Let $\tilde{g}$ be the metric tensor on $\mathbb{R}^n$ such that $\chi^*\tilde{g}$ is the usual metric tensor on $\mathbb{S}^n(r) - \{south\, pole\}$. Note that $\tilde{g}$ is conformally equivalent to the standard metric tensor $\bar{g}$ on $\mathbb{R}^n$, specifically,

$$\tilde{g}_p = r^2\left(\frac{2}{1+||p||^2}\right)^2 \bar{g}_p$$

at each $p \in \mathbb{R}^n$, where $||\ ||$ is the Euclidean norm in $\mathbb{R}^n$. Hence, if we denote the Levi–Civita connection of $\tilde{g}$ by $\tilde{\nabla}$, it can be shown that, at each $p \in \mathbb{R}^n$,

$$\tilde{\nabla} Z = \frac{1-||p||^2}{1+||p||^2}\, id.$$

Also by a straightforward computation, it can be shown that

$$(\tilde{\nabla}\tilde{\nabla} Z)(X, Y) = -\frac{1}{r^2}\tilde{g}(Z, X)Y$$

for all $X, Y \in \Gamma T(\mathbb{S}^n(r) - \{south\, pole\})$. Since $(\mathbb{S}^n(r) - \{south\, pole\}, \chi^*\tilde{g})$ and $(\mathbb{R}^n, \tilde{g})$ are isometric by the stereographic projection, the vector field on $\mathbb{S}^n(r)$, obtained by taking the lift of $Z$ to $\mathbb{S}^n(r) - \{south\, pole\}$ and defining its value as the zero vector at the south pole, also satisfies the above equation on $\mathbb{S}^n(r)$. Note that, this way we can construct $n+1$ linearly independent such vector fields on $\mathbb{S}^n(r)$.

**Remark 3.7.** Note from Lemma 3.1 and Proposition 3.3 that, on an $n(\geqslant 2)$-dimensional Riemannian manifold $(M, g)$, if a nonzero vector field $Z$ on $(M, g)$ satisfies the equation $(\nabla\nabla Z)(\cdot\,,\cdot) + \lambda g(Z,\cdot)(\cdot) = 0$, $\lambda \neq 0$, then $div\, Z$ satisfies the equation $\nabla\nabla div\, Z + \lambda(div\, Z)id = 0$ on $(M, g)$. Conversely, if a nonconstant function $f$ on an $n(\geqslant 2)$-dimensional Riemannian manifold $(M, g)$ satisfies the equation $H_f + \lambda f\, id = 0$, $\lambda \in \mathbb{R}$, then it can be easily shown that the vector field $Z = grad(f)$ satisfies the equation $(\nabla\nabla Z)(\cdot\,,\cdot) + \lambda g(Z,\cdot)(\cdot) = 0$ on $(M, g)$, where $div\, Z = -n\lambda f$. Hence, in this sense, the differential equations $H_f + \lambda f g = 0$ and $(\nabla\nabla Z)(\cdot\,,\cdot) + \lambda g(Z,\cdot)(\cdot) = 0$ may be considered equivalent on a Riemannian manifold if $\lambda \neq 0$.

**Remark 3.8.** Let $(M, g)$ be an $n(\geqslant 2)$-dimensional Riemannian manifold. Now, if we take the trace of the differential equation $H_f + \lambda f g = 0$ with respect to $g$ on $(M, g)$, where $f$ is a function on $M$ and $H_f$ is the Hessian form of $f$, we obtain another differential equation (in fact, an eigenvalue equation) $\Delta f = -n\lambda f$ on $(M, g)$, where $\Delta f = div\, grad(f)$ is the Laplacian of $f$ on $(M, g)$. By a theorem of Obata ([11, Theorem 5]), it is known that, a necessary and sufficient condition for a connected, compact, Einstein $n(\geqslant 2)$-dimensional Riemannian manifold with constant scalar curvature $\tau > 0$ to be isometric with the Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$, is the existence of a nonconstant function $f$ on $M$ satisfying the differential equation $\Delta f + \frac{\tau}{n-1} f = 0$. That is, the differential equation $\Delta f + \frac{\tau}{n-1} f = 0$ has a nontrivial solution on a connected, compact, Einstein $n(\geqslant 2)$-dimensional Riemannian manifold with constant scalar curvature $\tau > 0$ if and only if the domain Riemannian manifold $(M, g)$ is the Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$. Hence, the differential equation $\Delta f + \frac{\tau}{n-1} f = 0$ on connected, compact, Einstein $n(\geqslant 2)$-dimensional Riemannian manifolds $(M, g)$ with constant scalar curvature $\tau > 0$ serves as an analytic characterization (or representative) of Euclidean spheres. (This can also be viewed as a spectral characterization of Euclidean spheres [11]. But from the viewpoint of this paper, we consider it as a differential equation characterization). It is also shown in [11] that, if a nonconstant function $f$ satisfies the differential equation $\Delta f + \frac{\tau}{n-1} f = 0$ on a connected, compact, Einstein $n(\geqslant 2)$-dimensional Riemannian manifold $(M, g)$ with constant scalar curvature $\tau > 0$ (then $(M, g)$ is the Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$), then $f$ also satisfies the differential equation $H_f + \frac{\tau}{n(n-1)} fg = 0$ on $(M, g)$. Intuitively, this means that the differential equation $H_f + \lambda fg = 0$, $\lambda > 0$, may be considered to be the unfolded version of the equation $\Delta f + n\lambda f = 0$, $\lambda > 0$, on a Riemannian manifold $(M, g)$ isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$. In fact, on the $n(\geqslant 2)$-dimensional Euclidean sphere of sectional curvature $\lambda (> 0)$, the eigenfunctions of the Laplacian $\Delta$ on the space of functions corresponding to the largest eigenvalue $-n\lambda$, are the solutions of the above two equations. Furthermore, the gradients of these eigenfunctions are certain special conformal vector fields given as in Proposition 3.3(c). A similar situation also exists the for the differential equation analyzed in this paper. Let $(M, g)$ be an $n(\geqslant 2)$-dimensional Riemannian manifold. If we take the trace of the equation $(\nabla\nabla Z)(\cdot, \cdot) + \lambda g(Z, \cdot)(\cdot) = 0$, $\lambda \in \mathbb{R}$, with respect to $g$ on $(M, g)$, where $Z$ is a vector field on $(M, g)$, we obtain another differential equation (in fact, an eigenvalue equation) $\Delta Z + \lambda Z = 0$ on $(M, g)$. By Theorem 3.10 in [6], it is known that, a necessary and sufficient condition for a connected, compact, Einstein $n(\geqslant 3)$-dimensional Riemannian manifold $(M, g)$ with scalar curvature $\tau > 0$ to be isometric with the Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$, is the existence of a nonzero vector field $Z$ on $(M, g)$ satisfying the differential equation $\Delta Z + \frac{\tau}{n(n-1)} Z = 0$. That is, the differential equation $\Delta Z + \frac{\tau}{n(n-1)} Z = 0$ has a nontrivial solution on a connected, compact, Einstein $n(\geqslant 3)$-dimensional Riemannian manifold with scalar curvature $\tau > 0$ if and only if the domain Riemannian manifold $(M, g)$ is the Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$. Hence, the differential equation $\Delta Z + \frac{\tau}{n(n-1)} Z = 0$, $\lambda > 0$, on connected, compact, Einstein $n(\geqslant 3)$-dimensional Riemannian manifolds $(M, g)$ with scalar curvature $\tau$, also serves as an analytic characterization (or representative) of Euclidean spheres. (This can also be viewed as a spectral characterization of Euclidean spheres [6]). It can be shown by using the results in [6] that, if a nonzero vector field $Z$ satisfies the differential equation $\Delta Z + \frac{\tau}{n(n-1)} Z = 0$ on a connected, compact, Einstein $n(\geqslant 3)$-dimensional Riemannian manifold $(M, g)$ with scalar curvature $\tau > 0$ (then $(M, g)$ is the Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$), then $Z$

also satisfies the differential equation $(\nabla\nabla Z)(\cdot\,,\cdot) + \frac{\tau}{n(n-1)} g(Z,\cdot)(\cdot) = 0$ on $(M,g)$. Intuitively, this means that the differential equation $(\nabla\nabla Z)(\cdot\,,\cdot) + \lambda g(Z,\cdot)(\cdot) = 0$, $\lambda > 0$, may be considered to be the unfolded version of the equation $\Delta Z + \lambda Z = 0$, $\lambda > 0$, on a Riemannian manifold $(M,g)$ isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$. In fact, on the $n(\geqslant 2)$-dimensional Euclidean sphere of sectional curvature $\lambda(>0)$, the gradients of the eigenfunctions of the Laplacian $\Delta$ on the space of functions corresponding to the largest eigenvalue $-n\lambda$, are the solutions of the above two equations. Furthermore, the gradients of these eigenfunctions are certain special conformal vector fields given as in Proposition 3.3(c). Obata also stated another differential equation (see [12,14]) on connected, complete $n(\geqslant 2)$-dimensional Riemannian manifolds $(M,g)$ characterizing the Euclidean spheres by the existence of a nonconstant function $f$ on $M$ satisfying the equation $(\nabla\nabla grad(f))(X,Y) + \lambda[2g(grad(f),X)Y + g(Y,grad(f))X + g(X,Y)grad(f)] = 0$, $\lambda > 0$, for all $X, Y \in \Gamma TM$, that is, a necessary and sufficient condition for a connected, complete $n(\geqslant 2)$-dimensional Riemannian manifold $(M,g)$ to be isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$, $\lambda > 0$, is the existence of a nonconstant function $f$ on $M$ satisfying this equation. In fact, without loss of generality, this characterization of Euclidean spheres can be considered as a characterization by the existence of a nonzero vector field $Z$ on $(M,g)$ satisfying the equation $(\nabla\nabla Z)(X,Y) + \lambda[2g(Z,X)Y + g(Y,Z)X + g(X,Y)Z] = 0$, $\lambda > 0$, for all $X, Y \in \Gamma TM$. (Indeed, it can be shown as in Lemma 3.1 that, if a nonzero vector field $Z$ satisfies this equation then $\nabla div\, Z = -2\lambda(n+1)Z$). It is shown in [14] that, a nonzero vector field $Z$ satisfies this equation on a connected, $n(\geqslant 2)$-dimensional Riemannian manifold $(M,g)$ if and only if $Z$ is a $\lambda$-nullity and projective vector field on $(M,g)$. (See [14, Proposition 2.1]). In fact, on the $n(\geqslant 2)$-dimensional Euclidean sphere of sectional curvature $\lambda(>0)$, the vector fields satisfying this equation are the gradients of the eigenfunctions of the Laplacian $\Delta$ on the space of functions corresponding to the second largest eigenvalue $-2\lambda(n+1)$ of the Laplacian. Consequently, the equation $(\nabla\nabla Z)(X,Y) + \lambda g(Z,X)Y = 0$, $\lambda > 0$ (recall Remark 3.7), may also be considered to be the counterpart of the equation $(\nabla\nabla Z)(X,Y) + \lambda[2g(Z,X)Y + g(Y,Z)X + g(X,Y)Z] = 0$, $\lambda > 0$, related with the conformal geometry of Euclidean spheres, while the latter one may be considered to be related with the projective geometry of Euclidean spheres. In turn, perhaps the largest and the second largest eigenvalues of the Laplacian on Euclidean spheres may be considered to be related to the conformal and projective geometries of Euclidean spheres, respectively.

Unfortunately, from the viewpoint of this paper, the differential equation $(\nabla\nabla Z)(\cdot\,,\cdot) + \lambda g(Z,\cdot)(\cdot) = 0$, $\lambda \leqslant 0$, on a connected, complete Riemannian manifold $(M,g)$ is not that deterministic about its domain as in the case of $\lambda > 0$. In fact, the situation is the same for the accompanying differential equation $H_f + \lambda f g = 0$, $\lambda \leqslant 0$, on a complete Riemannian manifold $(M,g)$. (See [9,15]). Next we state the analogs of Theorem 3.5 for the case $\lambda \leqslant 0$. The proofs of these theorems can be obtained similarly as in the proof of Theorem 3.5. The only difference is to use corresponding results in [9] or [15] for the differential equation $H_f + \lambda f g = 0$, $\lambda \leqslant 0$, instead of the Obata's result for the equation $H_f + \lambda fg = 0$, $\lambda > 0$, on a connected, complete Riemannian manifold $(M, g)$.

First we analyze the case $\lambda = 0$ for the differential equation $(\nabla\nabla Z)(\cdot, \cdot) + \lambda g(Z, \cdot)(\cdot) = 0$ on a connected, complete Riemannian manifold $(M, g)$. Unfortunately, this case of the above equation is rather inconclusive in determining the domain Riemannian manifolds even in the case of existence of a nontrivial solution $Z$ (to the equation $\nabla\nabla Z = 0$). The reason is, if a nonzero vector field $Z$ satisfies the equation $\nabla\nabla Z = 0$ on a connected Riemannian manifold $(M, g)$, then by Lemma 3.1, *div* $Z$ is a constant function on $M$. Thus, we cannot use either Theorem B of [9] or Theorem 2-(I,A) of [15] to obtain at least a decomposition result for $(M, g)$. Moreover, even $\nabla Z$ may not be a self-adjoint bundle homomorphism on $(M, g)$. Eventually, we have only the following well-known result for this case.

**Theorem 3.9.** *Let $(M, g)$ be a connected, compact $n(\geqslant 2)$-dimensional Riemannian manifold. Then, a vector field $Z$ on $(M, g)$ is a solution of the equation $\nabla\nabla Z = 0$ on $(M, g)$ if and only if $Z$ is parallel on $(M, g)$.*

**Proof.** See [16, p. 39]. □

Now we analyze the case $\lambda < 0$ for the differential equation $(\nabla\nabla Z)(\cdot, \cdot) + \lambda g(Z, \cdot)(\cdot) = 0$ on a connected, complete Riemannian manifold $(M, g)$. In the theorem below, we use Theorem C of [9] or Theorem 2-(II,B) of [15] for the equation $H_f + \lambda fg$, $\lambda < 0$, instead of Obata's theorem for the equation $H_f + \lambda fg = 0$, $\lambda > 0$. That is, a necessary and sufficient condition for a connected, complete Riemannian manifold $(M, g)$ to be isometric with a connected component of the hyperbolic space of pseudo-radius $1/\sqrt{-\lambda}$, $\lambda < 0$, is the existence of a nonconstant function $f$ on $M$ satisfying the equation $H_f + \lambda fg = 0$ with $grad(f)(p) = 0$ at some $p \in M$.

**Theorem 3.10.** *Let $(M, g)$ be a connected, complete $n(\geqslant 2)$-dimensional Riemannian manifold. Then, a necessary and sufficient condition for $(M, g)$ to be isometric with a connected component of the hyperbolic space of pseudo-radius $1/\sqrt{-\lambda}$, $\lambda < 0$, is the existence of a nonzero vector field $Z$ on $(M, g)$ with $Z_p = 0$ at some $p \in M$ and satisfying the equation*

$$(\nabla\nabla Z)(X, Y) + \lambda g(Z, X)Y = 0$$

*for all $X, Y \in \Gamma TM$.*

Although the above theorem seems to indicate that a connected component of the hyperbolic space $\mathbb{H}^n_{\pm}(r)$ of pseudo-radius $r = 1/\sqrt{-\lambda}$, $\lambda < 0$, serves as a domain Riemannian manifold for the existence of nontrivial solutions $Z \in \Gamma TM$ to the differential equation $(\nabla\nabla Z)(\cdot, \cdot) + \lambda g(Z, \cdot)(\cdot) = 0$, $\lambda < 0$, it is not of much interest from the viewpoint of this paper because it depends on the existence of a particular solution $Z$ to this differential equation with $Z_p = 0$ at some $p \in M$.

We give an example of the necessary part of the theorem above in the following remark.

**Remark 3.11.** Now, we give an example of a vector field $Z$ on a connected component of the $n(\geqslant 2)$-dimensional hyperbolic space $(\mathbb{H}^n_{\pm}(r), \tilde{g})$ of pseudo-radius $r = 1/\sqrt{-\lambda}$, $\lambda < 0$, which satisfies the equation $(\tilde{\nabla}\tilde{\nabla}Z)(X, Y) + \lambda\tilde{g}(Z, X)Y = 0$ for all $X, Y \in \Gamma T\mathbb{H}^n_{\pm}(r)$, where $\tilde{\nabla}$ is the Levi–Civita connection of $(\mathbb{H}^n_{\pm}(r), \tilde{g})$. Now consider $(\mathbb{H}^n_{\pm}(r), \tilde{g})$ geometrically as the Poincaré disc: $D^n = \{p \in \mathbb{R}^n \mid ||p|| < 1\}$ with metric tensor

$$\tilde{g}_p = r^2\left(\frac{2}{1 - ||p||^2}\right)^2 \bar{g}_p$$

at each $p \in \mathbb{R}^n$, where $|| \ ||$ is the Euclidean norm in $\mathbb{R}^n$ and $\bar{g}$ is the Euclidean metric tensor on $D^n$ (see [8, p. 56] for details). Then the vector field $Z$ on $D^n$ defined by $Z_p = (p, p)$ has the property that

$$\tilde{\nabla}Z = \frac{1 + ||p||^2}{1 - ||p||^2}\, id$$

at each $p \in \mathbb{R}^n$. Hence, it can be shown by a straightforward computation that $Z$ satisfies the equation

$$(\tilde{\nabla}\tilde{\nabla}Z)(X, Y) = \frac{1}{r^2}\tilde{g}(Z, X)Y$$

for all $X, Y \in \Gamma TD^n$ with $Z_0 = 0$, where $0 \in D^n$.

The theorem below describes the remaining case from Theorem 3.10. In the proof of the theorem below, we use Theorem D of [9] for the equation $H_f + \lambda fg = 0$, $\lambda < 0$. That is, a necessary and sufficient condition for a connected, complete Riemannian manifold $(M, g)$ to be isometric with the warped product of the Euclidean line and a complete Riemannian manifold, where warping function $\psi$ on $\mathbb{R}$ satisfies the equation $\frac{d^2\psi}{dt^2} + \lambda\psi = 0$, $\psi > 0$, is the existence of a nonconstant function $f$ on $M$ satisfying the equation $H_f + \lambda fg = 0$, $\lambda < 0$, with $grad(f)(p) \neq 0$ at each $p \in M$.

**Theorem 3.12.** *Let $(M, g)$ be a connected, complete $n(\geqslant 2)$-dimensional Riemannian manifold and $\lambda \in (-\infty, 0)$. Then, a necessary and sufficient condition for $(M, g)$ to be isometric with the warped product of the Euclidean line and a complete Riemannian manifold, where warping function $\psi$ on $\mathbb{R}$ satisfies the equation $\frac{d^2\psi}{dt^2} + \lambda\psi = 0$, $\psi > 0$, is the existence of a nonvanishing vector field $Z$ on $(M, g)$ satisfying the equation*

$$(\nabla\nabla Z)(X, Y) + \lambda g(Z, X)Y = 0$$

*for all $X, Y \in \Gamma TM$.*

We give an example of the necessary part of the above theorem in the remark below.

**Remark 3.13.** It is known from Corollary 2 of [9] that, if $(M, g)$ is a warped product of the Euclidean line and a Riemannian manifold with warping function $\psi$ on $\mathbb{R}$ satisfying $\frac{d^2\psi}{dt^2} + \lambda\psi = 0$, $\psi > 0$, then there exists a function $f$ on $M$ satisfying the equation $H_f + \lambda f\, id = 0$ without critical points, where $\lambda < 0$. If we set $Z = grad(f)$, then it is easy to see that $Z$ satisfies the equation $(\nabla\nabla Z)(X, Y) + \lambda g(Z, \cdot)(\cdot) = 0$ on $(M, g)$ with $Z_p \neq 0$ at each $p \in M$.

Note that, a connected component of the hyperbolic space of pseudo-radius $1/\sqrt{-\lambda}$, $\lambda < 0$, can be written as a warped product of the Euclidean line and the Euclidean space of dimension $n - 1$ with the warping function $\psi$ on $\mathbb{R}$ is given by $\psi(t) = e^{\pm\sqrt{-\lambda}t}$. Hence, by combining Theorems 3.10 and 3.12, we obtain the following theorem which may be of some interest in characterizing certain warped product Riemannian manifolds $(M, g)$.

**Theorem 3.14.** *Let $(M, g)$ be a connected, complete $n(\geqslant 2)$-dimensional Riemannian manifold and $0 > \lambda \in \mathbb{R}$. A necessary and sufficient condition for $(M, g)$ to be isometric with a warped product of the Euclidean line and a complete Riemannian manifold, where warping function $\psi$ on $\mathbb{R}$ satisfies the equation $\frac{d^2\psi}{dt^2} + \lambda\psi = 0$, $\psi > 0$, is the existence of a nonzero vector field $Z$ on $(M, g)$ satisfying the equation*

$$(\nabla\nabla Z)(X, Y) + \lambda g(Z, X)Y = 0$$

*for all $X, Y \in \Gamma TM$.*

**Remark 3.15.** Note that, the differential equation $(\nabla\nabla Z)(\cdot, \cdot) + \lambda g(Z, \cdot)(\cdot) = 0$, $\lambda < 0$, on connected, $n(\geqslant 2)$-dimensional Riemannian manifolds may be considered as a characterization of the Riemannian warped products of the Euclidean line and a complete Riemannian manifold with warping function $\psi$ on $\mathbb{R}$ satisfying the equation $\frac{d^2\psi}{dt^2} + \lambda\psi = 0$, $\psi > 0$.

## References


[1] D.V. Alekseevsky, S. Marchiafava, Transformations of a quaternionic Kähler manifold, C. R. Acad. Sci. Paris Série I 320 (1995) 703–708.

[2] D.E. Blair, On the characterization of complex projective space by differential equations, J. Math. Soc. Japan 27 (1) (1975) 9–19.

[3] Y.H. Clifton, R. Maltz, The $k$-nullity space of curvature operator, Michigan Math. J. 17 (1970) 85–89.

[4] L.P. Eisenhart, Riemannian Geometry, Princeton University Press, Princeton, NJ, 1950.

[5] F. Erkekoğlu, E. García-Río, D.N. Kupeli, B. Ünal, Characterizing specific Riemannian manifolds by differential equations, Acta Appl. Math. 76 (2003) 195–219.

[6] F. Erkekoğlu, D.N. Kupeli, B. Ünal, Some Results Related to the Laplacian on Vector Fields, preprint.

[7] D. Ferus, Totally geodesic foliations, Math. Ann. 188 (1970) 313–316.
[8] S. Gallot, D. Hulin, J. Lafontaine, Riemannian Geometry, Springer, Heidelberg, 1993.
[9] M. Kanai, On a differential equation characterizing a Riemannian structure of a manifold, Tokyo J. Math. 6 (1) (1983) 143–151.
[10] R. Molzon, K. Pinney Mortensen, A characterization of complex projective space up to biholomorphic isometry, J. Geom. Anal. 7 (4) (1997) 611–621.
[11] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Japan 14 (3) (1962) 333–340.
[12] M. Obata, Riemannian manifolds admitting a solution of a certain system of differential equations, Proceedings of United States–Japan Seminar in Differential Geometry, Kyoto, 1965, pp. 101–114.
[13] W.A. Poor, Differential Geometric Structures, McGraw-Hill, New York, 1981.
[14] S. Tanno, Some differential equations on Riemannian manifolds, J. Math. Soc. Japan 30 (3) (1978) 509–531.
[15] Y. Tashiro, Complete Riemannian manifolds and some vector fields, Trans. Amer. Math. Soc. 117 (1965) 251–275.
[16] K. Yano, Integral Formulas in Riemannian Geometry, Marcel Dekker, New York, 1970.